

Section 11.4 part 1

11.4 Splitting Fields

Let F be a field, $f \in F[x]$

If f does not factor in $F[x]$ into a product of linear factors, then we have an extension of F where f has an additional root

$$f = (x-u) f_1, \quad f_1 \in F(u)[x]$$

f_1 is a polynomial $\deg f_1 = \deg f - 1$

with coefficients in the field extension.

We continue the procedure (f_1 instead of f and $F(u)$ instead of F).

At the end, we produce a field extension $K \supset F$ such that

the polynomial $f \in F[x] \subset K[x]$ splits completely - factors into all linear factors.

Def For $f \in F[x]$ an extension $K \supset F$

is a splitting field of the polynomial f if:

$$(1) \quad f = c(x-u_1)\dots(x-u_n) \quad c, u_i \in K$$

$$f = pg, \quad p \text{-irreducible} \quad p, g \in F[x] \\ \deg p \geq 1$$

We can always construct an extension of F such that p has a root in the extension

$$F[x]/(p) = F(u) \supset F \\ u = [x] \text{ is a root of } p$$

$$(2) K = F(u_1, \dots, u_n)$$

The above procedure allows us to construct a splitting field for $f \in F[x]$:

If $L \supseteq F$ such that (1) is satisfied (i.e. $u_1, \dots, u_n \in L$), we

can take $L \supseteq F(u_1, \dots, u_n) \supseteq F$ in order to satisfy (2)

To summarize:

Th 11.13 For $f \in F[x]$, $\deg f = n > 0$, there exists a splitting field of f $K \supseteq F$ with $[K:F] \leq n!$

Q To which extend a polynomial $f \in F[x]$ determines its splitting field? A - up to an isomorphism

Th 11.14 Let $\sigma: F \rightarrow E$ be a field isomorphism.

Let $f \in F[x]$, $\deg f > 0$

Let K be a splitting field of $f \quad \{ \quad K \supseteq F$

- $L \xrightarrow{\sigma} E$ $\{ \quad \sigma f \in E[x]$

Notations from
Cor 11.8

$$f = a_0 + a_1 x + \dots + a_n x^n$$

$$\sigma f = \sigma(a_0) + \sigma(a_1)x + \dots + \sigma(a_n)x^n$$

Then σ extends to an isomorphism $K \cong L$

Classification:

1. " σ extends"

$$\begin{array}{ccc} K & \xrightarrow{\bar{\sigma}} & L \\ U & \cong & U \\ F & \xrightarrow[\sigma]{} & E \end{array} \quad \bar{\sigma}(a) = \sigma(a) \text{ if } a \in F$$

2. If $E = F$, and $\sigma: F \rightarrow F$ is the identity map then $\sigma f = f$.

In this case, the theorem claims that any two splitting fields of a polynomial f , K and L are isomorphic.

We have $K \supseteq F$, $L \supseteq F$.

Moreover, there exists an isomorphism $K \xrightarrow{\cong} L$
 $a \mapsto a$ for $a \in F$

for a proof, recall:

Cor 11.8 $\sigma: F \rightarrow E$ - an isomorphism of fields

$u \in K \supseteq F$, u algebraic over F with min poly $p \in F[x]$

$v \in L \supseteq F$, $v \xrightarrow{\exists \text{ min poly } p \in E[x]}$ $L \xrightarrow{\exists \text{ min poly } p \in E[x]}$

Then σ extends to $\bar{\sigma}: F(u) \xrightarrow{\cong} E(v)$

For induction step:

Assume that the theorem is true for all polynomials
of degree up to $n-1$ (every fields F, E and isomorphism σ)

Let $p \mid f$ be a monic irreducible factor of f

$$F[x] \ni p$$

$$E[x] \ni \sigma p$$

σp is also monic, irreducible
 $\sigma p \mid \sigma f$

K contains all roots of p

$$L \longrightarrow \longrightarrow \sigma p$$

Let $u \in K$ be one of these roots: $p(u) = 0$

$$v \in L \longrightarrow \longrightarrow \sigma p(v) = 0 \quad \left. \begin{array}{l} v \text{ is any root of } \sigma p \\ (\text{but not any root of } \sigma f) \end{array} \right\}$$

Cor 11.8 implies an isomorphism $\bar{\sigma}: F(u) \xrightarrow{\sim} E(v)$ - extension of σ

$$u \mapsto v$$

Now we have

$$f = (x-u) g, \quad g \in F(u)[x]$$

$$\deg g = \deg f - 1$$

$$\sigma f = (x - v) \bar{g}, \quad \bar{g} \in E(v)[x]$$

$$\begin{array}{ccc}
 K & \xrightarrow{\cong} & L \\
 \cup_1 & & \cup_1 \\
 F(u) & \xrightarrow[\sim]{\bar{\sigma}} & E(v) \\
 \cup_1 & & \cup_1 \\
 F & \xrightarrow[\sim]{\sigma} & E
 \end{array}
 \qquad
 \begin{array}{ll}
 g \in F(u)[x] & \bar{g} \in E(v)[x] \\
 f \in F[x] & \sigma f \in E[x]
 \end{array}$$

Now the construction of an isomorphism $K \cong L$ which extends σ reduces to the construction of an isomorphism $K \cong L$ which extends $\bar{\sigma}$.

Now the data is g and \bar{g} of degree one less than $\deg f$, and $\bar{\sigma}: F(u) \xrightarrow{\cong} E(v)$ instead of $\sigma: F \xrightarrow{\cong} E$.

We now can make use of the inductive assumption.